proposition holds: the motion $g(t, q)$ of system $G_n$ is periodic if and only if it is $L^-$-stable and the kinetic energy $T[g(t, q)]$ is periodic.

Thus the recurrence of the kinetic energy of the negatively Lagrange-stable system of $n$ bodies fully defines the character of the recurrence of the motion of the system. Since the differential equations (1.1) of the motion of the system of $n$ bodies have, in particular, the energy integral from which we can obtain an explicit expression for the force function $U$ of system $G_n$, it follows that all arguments and discussions can be applied to the function $U$. Such an approach may be convenient when the recurrence of the motions is checked experimentally, since when the masses are known, the force function depends on the distance between the bodies of the system.

*Conclusions.* 1°. The isochronism, i.e. the mutual comparability with respect to time recurrence of the kinetic energy and motion of the system of $n$ bodies is the necessary condition for the Lagrange stability of motion of such a system.

2°. The motion of an $n$-body system is determined by a $6n$-dimensional vector function, and its kinetic energy by a scalar function depending on the last $3n$ components of the velocities of motion of the system. The isochronism of the $6n$-dimensional vector function $g(t,q)$ and scalar functkon $T[g(t, q)] = T(t)$ is characterized by the fact that the Lagrange stability is a special property of the motion of an $n$-body system. Since the Lagrange stability represents one of the possible forms of stability of the motion, it is possible for the kinetic energy to be minimal in some sense along the trajectory of the Lagrange-stable motion of an $n$-body system. In particular, this is the case for the kinetic energy of a recurrent, almost periodic and periodic motion of an $n$-body system. In the cases discussed above the kinetic energy is minimal in the Birkhof sense.

3°. Energy constructions have a long history in celestial mechanics. However, this is apparently the first time that the energy integral and its corollaries have been applied directly to the qualitative study of the motion of an $n$-body system in the form given here. We also note that the basic results remain valid for other forms of interaction between bodies, provided that they depend on the distance only.

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# THE SEPARATION OF MOTIONS IN SYSTEMS WITH RAPIDLY ROTATING PHASE*

A.I. NEISHTADT

In different versions of the method of averaging, the motion is separated into rapid oscillations and a slow drift with an accuracy depending on the order of approximation. It is shown below that in analytic systems with rapidly rotating phase this separation can be achieved so that the error is exponentially small. The remaining small error is shown to be theoretically impossible to eliminate in any version of the averaging method. From the statement of exponentially exact separation of oscillations and drift it follows in particular that the time the adiabatic invariant is maintained in single-frequency Hamiltonian systems (such as a pendulum with a slowly varying frequency, a charged particle in a weakly inhomogeneous field, etc.) is exponentially large. This statement is also used to prove that the splitting of the separatrix that occurs in the neighbourhood of resonance close to an integrable Hamiltonian system with to degrees of freedom, is exponentially small.

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**1. Statement of the problem.** Systems with rapidly rotating phase represent one of the fundamental objects of the asymptotic theory of non-linear oscillations [1]. Thier equations of motion have the form

$$x^{\cdot} = \varepsilon f(x, \varphi, \varepsilon), \ \varphi^{\cdot} = \omega(x) + \varepsilon g(x, \varphi, \varepsilon) \qquad (1.1)$$
$$x \in R^n, \ \varphi \bmod 2\pi \in S^1, \ \varepsilon \in [0, \varepsilon_0]$$

Here $\varepsilon > 0$ is a small parameter, $x$ is an $n$-dimensional vector called the vector of slow variables, and $\varphi$ is a scalar angular variable called the phase, with the right sides of (1.1) $2\pi$-periodic in $\varphi$. It is assumed that the frequency $\omega(x)$ does not vanish in the region considered, i.e. $\omega(x) > c^{-1} > 0$, $c = \text{const}$.

The symptotic method of averaging enables us to construct for any integer $m > 0$ a $2\pi$-periodic in $\varphi$ close to identical replacement of variables $(x, \varphi) \to (y, \psi)$:

$$x = y + \varepsilon U(y, \psi, \varepsilon), \ \varphi = \psi + \varepsilon V(y, \psi, \varepsilon) \qquad (1.2)$$
$$U = O(1), \quad V = O(1)$$

so that the equations for new variables contain a phase $\psi$ only in terms of order $\varepsilon^{m+1}$

$$y^{\cdot} = \varepsilon(F(y, \varepsilon) + \alpha(y, \psi, \varepsilon)), \quad \psi^{\cdot} = \Omega(y, \varepsilon) + \varepsilon\beta(y, \psi, \varepsilon) \qquad (1.3)$$
$$\alpha = O(\varepsilon^m), \quad \beta = O(\varepsilon^m)$$
$$F = \langle f \rangle + O(\varepsilon), \ \Omega = \omega + O(\varepsilon)$$

The angle brackets denote averafing over the phase $\varphi$. The functions $U$, $V$ are polynomials of power $m$ in $\varepsilon$.

Neglecting on the right side of (1.3) the small terms $\alpha$, $\beta$, we obtain a shortened system of the $m$-th approximation whose solutions approximate the solutions of (1.3) and, also, of the input system (1.1) in the time interval $0 \leqslant t \leqslant 1/\varepsilon$ with accuracy $O(\varepsilon^m)$ for $x$ and $O(\varepsilon^{m-1})$ for $\varphi$. The shortened system is much simpler to investigate than the original, as in it the drift is completely separated from the rapid oscillations. For example, its integration step can be $1/\varepsilon$ times greater than for (1.1).

When $m$ approaches infinity, we obtain, for the replacement of variables (1.2), series in $\varepsilon$ which are, however, generally divergent. Hence it is not possible to eliminate the phase from the equations and separate the slow and rapid motions. We consider below the question of the limiting accuracy of the separation of motion. It turns out that it is possible to achieve an accuracy of

$$O(\exp(-c_1^{-1}/\varepsilon)), \quad c_1 = \text{const}$$

**2. Formulations.** Let the right side of (1.1) be real-analytic functions in the complex $\delta$-neighbourhood $D + \delta$ of the real region $D = G\{x\} \times S^1\{\varphi\}$, that satisfies the estimates

$$|f| < C, \quad |g| < C, \quad c^{-1} < |\omega| < C$$

where $\delta$, $c$, $C$ are constants and $G$ is a region in $R^n$.

*Theorem 1.* When $(y, \psi) \in D + {}^1/_2\delta$, $0 < \varepsilon < \varepsilon_1$, there exists a real analytic $2\pi$-periodic in $\psi$ replacement of variables of the form (1.2) that reduces the system to the form (1.3) with exponentially small $\alpha$ and $\beta$

$$|\alpha| + |\beta| < c_2 \exp(-c_1^{-1}/\varepsilon), \quad |U| + |V| < c_3 \qquad (2.1)$$
$$|F - \langle f \rangle| + |\Omega - \omega| < c_4\varepsilon$$

where $\varepsilon_1, c_1, \ldots, c_4$ are positive constants which depend on the constants $\varepsilon_0$, $\delta$, $c$, $C$ introduced above.

*Remark 1.* A more exact elimination of the fast variable from the equations is, generally, impossible. Sect.5 gives an example of a system in which for any replacement of variables of the form (1.2) an exponentially small term dependent on the phase remains.

*Remark 2.* If system (1.1) is Hamiltonian, the replacement (1.2) can be of canonical form.

*Remark 3.* Let $t$ play the part of phase, i.e. (1.1) has the form

$$x^{\cdot} = \varepsilon f(x, t, \varepsilon) \qquad (2.2)$$

Then Theorem 1 holds for $f$ continuous with respect to $t$ and is a real-analytic with respect to $x$. The replacement of variables has the form $x = y + \varepsilon U(y, t, \varepsilon)$ and the function $U$ is real-analytic in $y$, while the smoothness of $U$ with respect to $t$ is one greater than the smoothness of $f$. If system (2.2) is Hamiltonian, the replacement of variables can be selected to be of canonical form.

The mapping of sequence of system (2.2) in the time $t = 2\pi$ has the form

$$x' = x + \varepsilon l(x, \varepsilon) \qquad (2.3)$$

Conversely, any representation of the form (2.3) can, obviously, be obtained as the mapping of a sequence for some system of the form (2.2) with function $f(x, t, \varepsilon)$ continuous with respect to $t$. Hence Remark 3 implies the following theorem on the mapping (2.3).

*Theorem 2.* Let the function $l(x, \varepsilon)$ be real-analytic and satisfy the estimate $|l(x, \varepsilon)| < C$ when $x \in G + \delta$. Then, when $y \in G + {}^1/_2\delta$, $0 < \varepsilon < \varepsilon_1$, there exists a real-analytic separation of variables

$$x = y + \varepsilon U(y, \varepsilon), \quad |U| < c_3 \tag{2.4}$$

that reduces (2.3) to the form

$$y' = y + \varepsilon(L(y, \varepsilon) + \alpha(y, \varepsilon)), \quad |\alpha| < c_2 \exp(-c_1^{-1}/\varepsilon) \tag{2.5}$$

with shortened mapping (2.5) (with rejected $\alpha$) is included in the stream, i.e. it can be represented as the mapping of the displacement in the time $t = 2\pi$ for the autonomous analytic system

$$y^{\cdot} = \varepsilon F(y, \varepsilon), \quad |F - 2\pi l| < c_4\varepsilon \tag{2.6}$$

where $\varepsilon_1, c_1, \ldots, c_4$ are constants which depend on $\varepsilon_0, \delta, C$.

*Remark 4.* A more exact inclusion of mapping (2.3) in the stream is generally not possible (see Sect.5).

*Remark 5.* When the mapping (2.3) is canonical and smoothly depends on $\varepsilon$, the replacement (2.4) can be selected to be of canonical form, and system (2.6) of Hamiltonian form; its Hamiltonian $H(x, \varepsilon)$ satisfies the estimate

$$|H(x, \varepsilon) - E(x)| < c_5\varepsilon \tag{2.7}$$

$$E(x) = 2\pi \int_{x_0}^{x} l^{(q)}\, dp - l^{(p)}\, dq$$

where $q$, $p$ are conjugate components of $x$, and $l^{(q)}, l^{(p)}$ are their respective displacement components $l$, taken for $\varepsilon = 0$, and the integral in (2.6) is independent of the integration path by definition of the canonical transformation.

## 3. The use of the adiabatic invariant to estimate the preservation time.

Let system (1.1) be Hamiltonian and $I$ the slow variable conjugate to the phase $\varphi$. The variable $I$ is an adiabatic invariant [2]: in the time interval $0 \leqslant t \leqslant 1/\varepsilon$ it only undergoes oscillations of the order of $\varepsilon$, whereas the order of variation of the remaining slow variables can be of the order of unity. The following statement evaluates the variation of $I$ over an exponentially long time interval.

*Proposition 1.* If solution (1.1) does not leave the region $D$ when $0 \leqslant t < T = \exp({}^1/_2 c_1^{-1}/\varepsilon)$ then along that solution over this time interval the difference between $I$ and its value at $t = 0$ does not exceed $3c_3\varepsilon$. Here the constants $c_1, c_3$ are taken from Theorem 1.

*Proof.* According to Remark 2 of Sect.2 the replacement (1.2) can be selected to be of canonical form. Let $J$ be a varaible, conjugate to the new phase $\psi$. In the shortened system (1.3) (with rejected $\alpha$), $J$ is an integral since the Hamiltonian does not contain $\psi$. In the complete system (1.3) $J$ varies exponentially slowly $|J^{\cdot}| < c_2 \exp(-c_1^{-1}/\varepsilon)$ its variation during the time $T$ is exponentially small. Let $I_0$, $J_0$ be the values of $I$, $J$ when $t = 0$. Then $|I - I_0| \leqslant |I - J| + |J - J_0| + |J_0 - I_0| < 3c_3\varepsilon$, which it was required to prove.

*Example 1.* Consider a linear oscillator with a slow varying frequency $\xi^{\cdot\cdot} + \omega^2(\varepsilon t)\xi = 0$. Let the function $\omega(\tau)$ be analytic in the strip $|\operatorname{Im}\tau| < \delta$ and satisfy the estimates $c^{-1} < |\omega| < C$. Then over an exponentially large time interval the adiabatic invariant $I = E/\omega$, $E = {}^1/_2(\xi^{\cdot 2} + \omega^2\xi^2)$ (the ratio of the oscillator energy to its frequency [2]) undergoes only oscillations of order $\varepsilon$.

*Example 2.* The motion of a charged particle in a weakly non-uniform magnetic field is represented as the motion along the Larmor circle that dirfts along a line of force of the field. Let the relative field strength variation along the length of the Larmor radius not exceed $\varepsilon \ll 1$. We denote by $w_\perp$ the velocity component normal to the field, line of force of the and by $B$ the magnetic field. Over an exponentially large time interval the transverse adiabatic invariant $I = w_\perp^2/(2B)$ [1] is a quantity conjugate to the phase of the motion along the Larmor circle, undergoes only oscillations of order $\varepsilon$ (on the assumption that in that time interval the particle does not leave the region of the field considered and that the field is analytic).

As in Proposition 1, the statement relative to the Hamiltonian system (2.2) follows from Remark 3 of Sect.2. Let $\varepsilon H(x, t)$ be the Hamiltonian of such a system.

*Proposition 2.* If solution (2.2) does not leave region $G$ when $0 \leqslant t \leqslant \exp({}^1/_2 c_1^{-1}/\varepsilon)$, then along that solution in such a time interval the function $\langle H(x,\cdot)\rangle$ undergoes oscillations not exceeding $3c_3\varepsilon$.

*Example 3.* Let a point move in the rapidly oscillating one-dimensional analytic potential well: its energy is $H = {}^1/_2\,(p^2 + U(q,\nu))$, $\nu = t/\varepsilon$. In conformity with Proposition 2 the value of the function $\langle H\rangle = {}^1/_2\,(p^2 + \langle U(q,\cdot)\rangle)$ during an exponentially large time interval changes only by a quantity of order $\varepsilon$.

For the canonical mapping of the form (2.3) a similar statement follows from Remark 5 in Sect.2. Let the function $E(x)$ be defined in conformity with (2.7). The successive application of mapping provides the discrete trajectory of the point.

*Proposition 3.* If for $N = [\exp({}^1/_2 c_1{}^{-1}/\varepsilon)]$ iterations of the mapping (2.3) the point does not leave $G$, then the oscillations $E(x)$ on its trajectory do not exceed $3c_3\varepsilon$.

*Example 4.* Consider the motion of a small sphere between two walls the distance between which changes slowly. The collision of the sphere with the wall is assumed to be perfectly elastic. Let the distance of the walls from the origin of coordinates be specified by the functions $d_1(\tau), d_2(\tau)$, $\tau = \varepsilon t$, analytic in the strip $|\operatorname{Im}\tau| < \delta$, and satisfy the estimates $|d_1| < C$, $c^{-1} < |d_1 - d_2| < C$. We will assume for simplicity that the origin of coordinates always lies between the walls $(d_1 < 0 < d_2)$. Let the small sphere pass through the origin of coordinates at the instant $t$ with velocity $w$. On being reflected from the two walls it again passes through the origin of coordinates in the same direction with velcoity $w'$ at some instant $t'$.

It can be verified that the mapping $\varepsilon t$, ${}^1/_2 w^2 \to \varepsilon t'$, ${}^1/_2 w'^2$ is canonical and close to identical. Calculations show that the adiabatic invariant $I = w(d_2 - d_1)$ plays the part of the function $E$ (2.7) [2] (with an accuracy of $8\pi$). According to Proposition 3, $I$ can change over an exponentially long time interval not more that by the quantity $k_1\varepsilon$, $k_1 = \text{const}$.

*Example 5.* Let it now be assumed in the problem of Example 4 that the walls do not move slowly $d_1 = d_1(t)$, $d_2 = d_2(t)$. This problem was studied as a model in the analysis of particle acceleration induced by quasi-random collisions (the "Fermi-Ulam model" [3]). Let us estimate the upper limit of the rate of acceleration. We use the following reasoning here: if the velocity of the sphere becomes considerably greater than that of walls, then the adiabatic invariant $I = w(d_2 - d_1)$ occurs in the problem, and this inhibits rapid acceleration. Suppose that at some instant the sphere passes through the origin of coordinates having $I = I_0$ and let $\varepsilon = k/I_0$, $k = \sup_t(|d_1^{\cdot}|, |d_2^{\cdot}|)$. The substitution of the time $\nu = t/\varepsilon$ yields the problem of Example 2. Let $N$ be the minimum number of collisions with the walls, which yield $I > I_0 + k_1$. The analysis of Example 2 shows that $N > \exp(k_2 I_0)$, $k_2 = \text{const}$. The upper limit for the rate of acceleration is obtained on the assumption that after $\exp(k_2 I)$ collisions $I$ increases by $k_1$. Then for $n$ collisions we have $w = O(\ln n)$, i.e. only logarithmic acceleration is possible in this problem.

## 4. Proof of Theorem 1.

The necessary replacement of variables is carried out as in [4—6] in the form of a composition of a large number of successively determined substitutions that cancel the dependence on the phase in terms of ever higher order. Further calculations are aimed at showing that it is possible to make $r \sim 1/\varepsilon$ such replacements, in which case terms that depend on the phase decrease at least in a geometric progression. Here, the estimates in [5] are applicable to system (1.1).

*4.1. Procedure for successive substitutions.* The system obtained after $i$ substitutions has the form

$$x^{\cdot} = \varepsilon(F_i(x) + \alpha_i(x,\ \varphi)),\quad \varphi^{\cdot} = \Omega_i(x) + \varepsilon\beta_i(x,\ \varphi) \tag{4.1}$$
$$\langle\alpha_i\rangle = \langle\beta_i\rangle = 0,\quad (x,\ \varphi) \in D_i,\ D + \delta/2 \subset D_i \subseteq D + \delta$$

The explicit dependence of the right side on $\varepsilon$ is not indicated here, $F_0 = \langle f\rangle$, $\alpha_0 = f - \langle f\rangle$, $\Omega_0 = \omega + \varepsilon\langle g\rangle$, $\beta_0 = g - \langle g\rangle$, $D_0 = D + \delta$. At the $(i+1)$-th step the replacement of variables is sought in the form

$$x = y + \varepsilon u(y,\ \varphi),\quad \varphi = \psi + \varepsilon v(y,\ \psi) \tag{4.2}$$

Substitution of (4.2) into (4.1) gives

$$y^{\cdot} = \varepsilon\Big(E + \varepsilon\frac{\partial u}{\partial y}\Big)^{-1}\Big(F_i(y + \varepsilon u) + \alpha_i(y + \varepsilon u, \varphi) - \frac{\partial u}{\partial\varphi}\Omega_i(y + \varepsilon u) - \varepsilon\frac{\partial u}{\partial\varphi}\beta_i(y + \varepsilon u, \varphi)\Big) = \varepsilon f_{i+1}(y, \varphi) \tag{4.3}$$
$$\psi^{\cdot} = \Big(1 + \varepsilon\frac{\partial v}{\partial\psi}\Big)^{-1}\Big(\Omega_i(y + \varepsilon u) + \varepsilon\beta_i(y + \varepsilon u, \psi + \varepsilon v) - \varepsilon^2\frac{\partial v}{\partial y}f_{i+1}(y, \varphi)\Big)$$

where $\varphi$ must be expressed in terms of $\psi, y$ in accordance with (4.2), and $E$ is the unit matrix of dimensions $n \times n$. The functions $u, v$ are selected so as to suppress the dependence of the right sides of (4.3) on $\psi$ in the principal order of $\varepsilon$

$$u(y,\varphi)=\frac{1}{\Omega_i(y)}\int_0^{\varphi}\alpha_i(y,\gamma)\,d\gamma \tag{4.4}$$

$$v(y,\varphi)=\frac{1}{\Omega_i(y)}\int_0^{\varphi}\Big(\beta_i(y,\gamma)+\frac{\partial\Omega_i}{\partial y}(u(y,\gamma)-\langle u\rangle)\Big)d\gamma$$

where $u$ and $v$ can be supplemented by arbitrary functions of $y$.

Let $\varepsilon a$, $\varepsilon b$ be the set of terms in the first and second equations (4.3) respectively, of order higher than the first in $\varepsilon$. Then

$$F_{i+1}=F_i+\langle a\rangle,\ \Omega_{i+1}=\Omega_i+\varepsilon\langle b\rangle+\varepsilon\frac{\partial\Omega_i}{\partial y}\langle u\rangle \tag{4.5}$$

$$\alpha_{i+1}=a-\langle a\rangle,\ \beta_{i+1}=b-\langle b\rangle$$

*4.2. Estimates.* Assume that $r$ steps of the procedure in Sect.4.1 have been made. The region $D_i$ in which the system is considered after $i$ substitutions is specified as $D_i=D_1-2(i-1)K\varepsilon$, $D_1=D+{}^3/_4\delta$, and $K$ is a positive constant that is defined below.

Considering the first step, we can show that formulas (4.2) and (4.4) for $y,\psi\in D_1$ and fairly small $\varepsilon$ in fact determine the replacement of variables, and

$$|\alpha_1|+|\beta_1|<k_1\varepsilon,\quad |u|+|v|<k_2\varepsilon \tag{4.6}$$

$$|F_1-\langle f\rangle|+|\Omega_1-\omega|<k_3\varepsilon$$

where $k_1$, $k_2$, $k_3$ and further $k_i$ are positive constants.

Let us use the inductive proposition that for $1\leqslant i\leqslant r$ the estimates

$$|F_i|<2C,\quad {}^1/_2c^{-1}<|\Omega_i|<2C \tag{4.7}$$

$$|\alpha_i|+|\beta_i|<M_i,\quad M_i=2^{-i+1}k_1\varepsilon$$

are satisfied.

We select the constants $\varepsilon_1$, $K$ so that the replacement of the variables (4.2) with $i=r$ is determined when $0<\varepsilon<\varepsilon_1$, $y\in D_{r+1}=D_r-2K\varepsilon$, and the equations obtained satisfy the estimates (4.7) with $i=r+1$ (it is of course assumed here that $D_{r+1}$ is non-empty).

It follows from (4.4), (4.7) and the Cauchy estimates [4] that for $i=1,\ldots,r$ and $y\in D_i-K\varepsilon$ the conditions

$$|\varepsilon u|<2c\pi M_i\varepsilon<k_4\varepsilon,\quad |\varepsilon v|<k_5\varepsilon \tag{4.8}$$

$$\left|\varepsilon\frac{\partial u}{\partial y}\right|<k_4/K,\left|\varepsilon\frac{\partial v}{\partial\psi}\right|<k_6\varepsilon$$

are satisfied.

We select $K>2\max\{k_4,k_5\}$, $\varepsilon_1<{}^1/_2k_6^{-1}$. Then the inequalities (4.8) for $u,v$ show that formulas (4.2) define the mapping of $D_{r+1}$ in $D_r-K\varepsilon$, and the inequalities for $\partial u/\partial y$, $\partial v/\partial y$ show that this mapping is a diffeomorphism of $D_{r+1}$, i.e. is in fact a definition of the replacement of variables. For the functions $a$ and $b$ (4.5) from (4.7) and (4.8) and the Cauchy estimates we have the estimates

$$|a|+|b|<k_7(K^{-1}+\varepsilon)M_i$$

Selecting $K$ fairly large and $\varepsilon$ fairly small, we obtain

$$|a|+|b|<{}^1/_4M_i,\quad |\alpha_{i+1}|+|\beta_{i+1}|<{}^1/_2M_i=M_{i+1} \tag{4.9}$$

$$|F_{i+1}-F_i|+|\Omega_{i+1}-\Omega_i|<{}^1/_4M_i$$

from which and (4.6) it follows that

$$|F_i-\langle f\rangle|+|\Omega_i-\omega|<k_8\varepsilon,\ i=1,\ \ldots,\ r+1$$

Hence for fairly small $\varepsilon$ the inductive inequalities (4.7) are satisfied for $i=r+1$. This means that the necessary replacements of variables with selected $K$ and $\varepsilon$ can be carried out as long as $D_r$ is non-empty. After $r=[{}^1/_4\delta K^{-1}/\varepsilon]>k_9/\varepsilon$ replacements of variables we have

$$|\alpha_r|+|\beta_r|<2^{-r+1}k_1\varepsilon<c_2\exp(-c_1^{-1}/\varepsilon)$$

and the remaining inequalities (2.1) are satisfied, which it was required to prove.

If system (1.1) is Hamiltonian, the substitutions in all approximations can be selected to be of canonical form. Their composition provides a canonical transformation whose existence is confirmed by Remark 2 in Sect.3.

If the time $t$ plays the part of the phase in system (1.1), then $v=0$ in (4.4) so that only $x$ is transformed. Hence for the estimates only the analyticity of $x$ on the right sides of the equations is necessary, and the continuity with respect to $t$ is sufficient, as stated in Remark 3 of Sect.2.

**5. The unimprovability of the estimates.** Consider the trivially integrable system of equations

$$x^{\cdot} = \varepsilon f(z, \varphi), \quad z^{\cdot} = \varepsilon, \quad \varphi^{\cdot} = 1 \tag{5.1}$$

$$f(z, \varphi) = \sin \varphi \sum_{m=1}^{\infty} \alpha_m \sin mz, \quad a_m = \exp(-m)$$

Let $z$ be the angular coordinate so that the replacements of variables must be selected as $2\pi$-periodic in $z$. When $\varphi$ is eliminated from (5.1), it is possible to leave out the transformation of $z, \varphi$ and consider replacements of the form $x = y + \varepsilon u(z, \varphi)$, whose substitution into (5.1) gives

$$y^{\cdot} = \varepsilon \left[ f(z, \varphi) - \frac{\partial u}{\partial \varphi} - \varepsilon \frac{\partial u}{\partial z} \right] = \varepsilon f_1(z, \varphi) \tag{5.2}$$

Let us estimate the lower bound of $|f_1 - \langle f_1 \rangle|$. For this we calculate for $\varepsilon = 1/m$ ($m$ is an integer) the integral over the closed path $\varphi - mz = \varphi_0 = \mathrm{const}$ on the torus $T^2 = \{(z, \varphi) \bmod 2\pi\}$

$$\int_0^{2\pi} (f_1(z, \varphi_0 + mz) - \langle f_1 \rangle)\, dz = \frac{1}{2} \exp\left(-\frac{1}{\varepsilon}\right) \cos \varphi_0$$

Hence when $\varepsilon = 1/m$ the inequality

$$\max |f_1(z, \varphi) - \langle f_1 \rangle| > (4\pi)^{-1} \exp(-1/\varepsilon)$$

is satisfied, i.e. for any replacement of variables an the right side of (4.2) there remains an exponentially small term dependent on the phase. This means that the estimate of Theorem 1 cannot be improved.

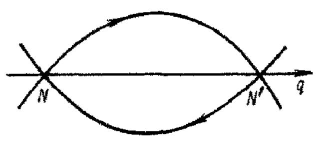


Fig.1

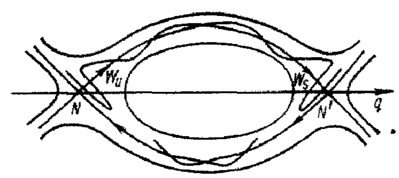


Fig.2

A similar reasoning for the mapping

$$x' = x + \varepsilon \sum_{m=1}^{\infty} a_m \sin mz, \quad z' = z + \varepsilon$$

proves the unimprovability of the estimates of Theorem 2.

**6. The exponential smallness of separatrix splitting.** We will use the statements of Sect.2 to estimate the magnitude of separatrix splitting which occurs close to the resonance in the Hamiltonian system with two degrees of freedom, close to integrable. When one of the angular variables is selected as the new time, the Hamiltonian of this system takes the form

$$H = H_0(I) + \mu H_1(I, \varphi, t) \tag{6.1}$$

where $I, \varphi$ are the conjugate canonical variables, $t$ is the time, $\mu > 0$ is a small parameter, and the Hamiltonian is $2\pi$-periodic in $\varphi, t$ and analytic in all variables. Let $I_*$ be the simple resonance value of $I$

$$\partial H_0(I_*)/\partial I = m/n, \quad \partial^2 H_0(I_*)/\partial I^2 = a \neq 0$$

where $m, n$ are relatively prime integers. Close to resonance we introduce new canonical variables $p = (I - I_*)/\sqrt{\mu}$, $q = \varphi - mt/n$, in which the Hamiltonian has the form

$$E = \sqrt{\mu}\,[{}^1\!/_2 ap^2 + V_1(q, t)] + \mu V_2(p, q, t, \mu) \tag{6.2}$$
$$V_1 = H_1(I_*, \varphi, t)$$

where $E$ has a period of $2\pi$ in $q$ and $2\pi n$ in $t$. If we neglect the last term in (6.2) and average the Hamiltonian over $t$, we obtain a system with one degree of freedom, which describes the motion of a point in the potential $U(q) = \langle V_1(q, \cdot) \rangle$ $2\pi/n$-periodic in $q$. Suppose $U(q)$ has a unique point of absolute maximum that is non-degenerate. Unstable equilibria of the averaged system, connected by separatrices (Fig.1), correspond to the maxima of $U(q)$. In the exact system (6.2) for fairly small $\mu$ to such a equilibria and separatrices there correspond unstable periodic solutions and surfaces asymptotic to them [7]. The cross sections $W_u$

and $W_s$ of these surfaces by the plane $t=0$ are shown in Fig.2. $W_u$ and $W_s$ are invariant curves of sequence mapping during the time $2\pi n$ for system (6.2) which are asymptotic to stationary points $N$ and $N'$. Curves $W_u$ and $W_s$ generally do not coincide with one another — the separatrices split. It was shown in [7] that finite segments of $W_u$ and $W_s$ are indistinguishable from one another with a power accuracy with respect to $\mu$ ; in existing examples the difference is of order $\exp(-\text{const}/\sqrt{\mu})$.

The following theorem shows that generally the splitting of separatrices is exponentially small.

*Theorem 3.* Curves $W_u$ and $W_s$ lie in the union $K_2\exp(-K_1^{-1}/\sqrt{\mu})$ of the neighbourhood of two analytic curves that are $2\pi/n$-periodic in $q$ and uniquely projected on the $q$ axis; $K_1$ and $K_2$ are positive constants.

*The scheme of proof.* According to Theorem 1 and Remark 3 to it, when $|p|<K_3$ there exists a diffeomorphism $\Phi: p, q \to P, Q$ (the replacement of variables), which is analytic, simplex, $2\pi n$ periodic in $t$ and $2\pi$ periodic in $q$, and close to identical, which reduces the Hamiltonian of the system to the form.

$$F = \sqrt{\phantom{\mu}}\,[{}^1/_2 aP^2 + U(Q)] + \mu R(P, Q, \mu) + \alpha(P, Q, t, \mu) \tag{6.3}$$
$$|R| = O(1),\quad |\alpha| = O(\exp(-K_4^{-1}/\sqrt{\mu}))$$
$$K_3 = 2(\max_q U - \min_q U) = \text{const},\ K_4 = \text{const}$$

where the function $R$ is $2\pi/n$ perodic in $Q$.

If the last term in (6.3) is neglected, a system with one degree of freedom is obtained. Its separatrices do not split, and as shown in Fig.1, form together with the singular points two analytic curves $S_1, S_2$, which divide the phase plane into regions. In accordance with [4] system (6.3) has a number of invariant tori whose sections by the plane $t=0$ are closed or periodic invariant curves of the mapping sequence (Fig.2). Using the estimates [8, 9] it can be shown that in each of the regions of subdivision of the phase surface there is an invariant curve lying at a distance $O(\exp(-K_1^{-1}/\sqrt{\mu}))$ from $S = S_1 \cup S_2$. These curves bound the invariant region $D$ in which the stationary points of the mapping sequence $\Phi(N), \Phi(N')$ and, also, their asymptotic invariant curves $\Phi(W_u), \Phi(W_s)$ lie. Hence $W_u$ and $W_s$ lie in $\Phi^{-1}(D)$ and, consequently, in the union $K_2\exp(-K_1^{-1}/\sqrt{\mu})$ of the neighbourhoods of the curves $\Phi^{-1}(S_1)$ and $\Phi^{-1}(S_2)$ , stated above.

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